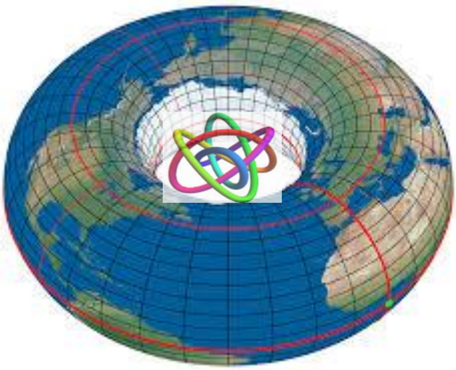
Topology:
A Special Way of Looking at the Spatial World



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by Dr C. D. H. Cooper
Macquarie University

These notes were prepared for students at Macquarie University in Australia but are freely available to anyone. However if you make use of them and are not a Macquarie University student it would be nice if you could email me at christopherdonaldcooper@gmail.com to let me know where you are from. And, if you are from outside of Australia perhaps you could send me a postcard of where you are from, to pin up on my wall (Christopher Cooper, 31 Epping Avenue, EASTWOOD, NSW 2122, Australia).

What is Topology?

Topology has many different guises. If we were interested in differential equations we might need to study what is known as ‘point-set topology’ where the concepts of topological spaces and continuous functions are defined very rigorously. Students find this very difficult and abstract, and unless they go on to see the applications they find it very uninspiring, for there are some very weird examples of topological spaces and only very special ones can be visualised.

These notes cover an excursion into what is known as ‘geometric topology’. Here we focus on those aspects of topology where we can visualise what is going on in some way. Students find it fascinating because it forces them to look at the spatial world from a new perspective.

Now geometry is also a study of the spatial world. However, while ordinary geometry is concerned with *quantitative* properties such as distances and angles and projective geometry is concerned with the property of collinearity, topology focuses on the way space is connected. It’s concerned with making precise the notions of nearness, connectedness and continuity.

But topology isn’t interested in the way the actual universe is connected. That’s a job for the cosmologist. Nor is it merely concerned with the way 3-dimensional

Euclidean space is connected. There are many possible 'spaces' that we can imagine – spaces where the concepts of nearness and continuity can be defined and spaces where small regions are connected just as they are in Euclidean space. It's up to the cosmologist to decide which one is the appropriate model for our universe. The topologist is much more interested in exploring what spaces are possible.

For example one alternative to the familiar 3-dimensional Euclidean space (and one that may be the correct model for our own universe) is a finite space where, if you go far enough in any direction, you will always return to where you started. This may seem strange, yet at the 2-dimensional level we're all familiar with the surface of a sphere where this is indeed the case.

After some general introductory material we make a deep study of surfaces. These include not only surfaces that we've all seen, such as the sphere, the cylinder and the torus (doughnut), but also some strange surfaces, such as the Projective Plane and the Klein Bottle, that can only ever exist in our imagination. And we don't merely discuss a few specific examples. We'll develop such an understanding of surfaces that we'll be able to classify them, that is, describe all possible surfaces.

One of the things that we'll do in order to classify surfaces is to 'draw' graphs (networks of points and edges) and maps on these surfaces. We will study questions of embeddability (can a given graph be drawn on a given surface so that the edges don't cross each other) and map colouring (what is the least number of colours needed to colour any map on a given surface if adjacent regions are to have different colours).

The second half of these notes focuses on the way 3-dimensional Euclidean space is connected, by studying knots. Imagine a knotted piece of string with the two ends glued together and you have a pretty good idea of what is meant by a knot in topology. Knots have been studied by chemists, physicists and biologists at different times because they've believed that the way molecules, hydrodynamic flow or DNA are knotted has a significance to their area of study. Indeed much of the pioneering work in Knot Theory was carried out by chemists, and some more recent developments were made by physicists.

Since the earliest days of Knot Theory the fundamental problem has been to find ways of deciding whether two given knots are equivalent, that is, whether one can be deformed into the other without cutting or untying. Well, actually it's the converse problem that is of more interest. If two knots are equivalent and we deform one, sooner or later we may end up with it looking like the other. If we manage to do this, even if it is a fluke, we will have proved that the two knots are equivalent. But if two knots are really different how do we show this? The fact that we cannot achieve a deformation from one to the other is no proof. Perhaps we need to spend more time, perhaps we need to be a bit cleverer!

The technique to show that two knots are inequivalent is to find invariants, numbers or other mathematical objects, which stay the same when we manipulate a knot but which are different for our two examples.

Many tools have been devised to assist with this problem, generally involving very deep mathematics that is studied only at postgraduate level. However the Alexander Number is sufficiently simple that a primary school child can compute it. Although they will not understand why it is an invariant they can be told that it is and so they can draw the conclusion that two given knots are inequivalent simply because they have different Alexander Numbers.

Of course few primary school children will be bothered asking why the Alexander Number works in distinguishing knots – which is fortunate because to do that they'd need to dig down to the level of the group theory behind it! But, in these notes, we do just that. We develop the Alexander Group of a knot. For those who've never studied Group Theory before, one chapter of these notes is devoted to giving an introduction to the subject.

Finally we introduce an ‘indeterminate’ so as to arrive at the Alexander Module and from that we reach the classic knot invariant, the Alexander Polynomial. This is a more powerful invariant that can be used to distinguish inequivalent knots that happen to have the same Alexander Number. An appendix lists all knots up to seven crossings.

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PART B: KNOTS AND LINKS